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LIFSHITZ TAILS FOR ALLOY TYPE MODELS IN A CONSTANT MAGNETIC FIELD

FRÉDÉRIC KLOPP

Dedicated to the memory of Pierre Duclos.

ABSTRACT. In this paper, we study Lifshitz tails for a 2D Landau Hamiltonian perturbed by a random alloy-type potential constructed with single site potentials decaying at least at a Gaussian speed. We prove that, if the Landau level stays preserved as a band edge for the perturbed Hamiltonian, at the Landau levels, the integrated density of states has a Lifshitz behavior of the type $e^{-\log^2 |E-2bq|}$.

RÉSUMÉ. Dans cette note, nous démontrons qu'en dimension 2, la densité d'états intégrée d'un opérateur de Landau avec un potentiel aléatoire non négatif de type Anderson dont le potentiel de simple site décroît au moins aussi vite qu'une fonction gaussienne admet en chaque niveau de Landau, disons, $2bq$, si celui-ci est un bord du spectre, une asymptotique de Lifshitz du type $e^{-\log^2 |E-2bq|}$.

0. INTRODUCTION

On $C_0^\infty(\mathbb{R}^2)$, consider the Landau Hamiltonian

$$H_0 = H_0(b) := (-i\nabla - A)^2 - b$$

where $A = (-\frac{bx_2}{2}, \frac{bx_1}{2})$ is the magnetic potential, and $b > 0$ is the constant scalar magnetic field. H_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$. It is well-known that $\sigma(H_0)$, the spectrum of the operator H_0 , consists of the so-called Landau levels $\{2bq; q \in \mathbb{N} = \{0, 1, 2, \dots\}\}$; each Landau level is an eigenvalue of infinite multiplicity of H_0 .

Consider now the random \mathbb{Z}^2 -ergodic alloy-type electric potential

$$V_\omega(x) := \sum_{\gamma \in \mathbb{Z}^2} \omega_\gamma u(x - \gamma), \quad x \in \mathbb{R}^2$$

where we assume that

- **H₁**: The single-site potential u satisfies, for some $C > 0$ and $x_0 \in \mathbb{R}^2$,

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$$\frac{1}{C} \mathbf{1}_{\{x \in \mathbb{R}^2; |x-x_0| < 1/C\}} \leq u(x) \leq C e^{-|x|^2/C}.$$

- **H₂**: The coupling constants $\{\omega_\gamma\}_{\gamma \in \mathbb{Z}^2}$ are non-trivial, almost surely bounded i.i.d. random variables.

These two assumptions guarantee V_ω is almost surely bounded. On the domain of H_0 , define the operator $H_\omega := H_0 + V_\omega$. The integrated density of states (IDS) of the operator H_ω is defined as the non-decreasing left-continuous function $\mathcal{N} : \mathbb{R} \rightarrow [0, \infty)$ which, almost surely, satisfies

$$\int_{\mathbb{R}} \varphi(E) d\mathcal{N}(E) = \lim_{R \rightarrow \infty} R^{-2} \text{Tr} (\mathbf{1}_{\Lambda_R} \varphi(H) \mathbf{1}_{\Lambda_R}), \quad \forall \varphi \in C_0^\infty(\mathbb{R}).$$

Here and in the sequel, $\Lambda_R := (-\frac{R}{2}, \frac{R}{2})^2$ and $\mathbf{1}_{\mathcal{O}}$ denotes the characteristic function of the set \mathcal{O} .

By the Pastur-Shubin formula (see e.g. [9, Section 2]), we have

$$\int_{\mathbb{R}} \varphi(E) d\mathcal{N}(E) = \mathbb{E} (\text{Tr} (\mathbf{1}_{\Lambda_1} \varphi(H) \mathbf{1}_{\Lambda_1})), \quad \forall \varphi \in C_0^\infty(\mathbb{R}),$$

where \mathbb{E} denotes the mathematical expectation with respect to the random variables $(\omega_\gamma)_\gamma$. Moreover, there exists a set $\Sigma \subset \mathbb{R}$ such that $\sigma(H_\omega) = \Sigma$ almost surely. Σ is the support of the positive measure $d\mathcal{N}$. The aim of the present article is to study the asymptotic behavior of \mathcal{N} near the edges of Σ . It is well known that, for many random models, this behavior is characterized by a very fast decay which goes under the name of “Lifshitz tails”. It was studied extensively (see e.g. [6, 9, 4] and references therein).

In order to fix the picture of the almost sure spectrum $\sigma(H_\omega)$, we assume:

- **H₃**: the common support of the random variables $(\omega_\gamma)_{\gamma \in \mathbb{Z}^2}$ consists of the interval $[\omega_-, \omega_+]$ where $\omega_- < \omega_+$ and $\omega_- \omega_+ = 0$.
- **H₄**: $M_+ - M_- < 2b$ where

$$\pm M_\pm := \text{ess-sup}_\omega \sup_{x \in \mathbb{R}^2} (\pm V_\omega(x)).$$

Assumptions **H₁** – **H₄** imply that $M_- M_+ = 0$. It also implies that the union $\bigcup_{q=0}^{\infty} [2bq + M_-, 2bq + M_+]$, which contains Σ , is disjoint. Let W be the bounded \mathbb{Z}^2 -periodic potential defined by

$$W(x) := \sum_{\gamma \in \mathbb{Z}^2} u(x - \gamma), \quad x \in \mathbb{R}^2.$$

On the domain of H_0 , define the operators $H^\pm := H_0 + \omega_\pm W$. It is easy to see that

$$\sigma(H^-) \subseteq \cup_{q=0}^{\infty} [2bq + M_-, 2bq], \quad \sigma(H^+) \subseteq \cup_{q=0}^{\infty} [2bq, 2bq + M_+],$$

and

$$\sigma(H^-) \cap [2bq + M_-, 2bq] \neq \emptyset, \quad \sigma(H^+) \cap [2bq, 2bq + M_+] \neq \emptyset, \quad \forall q \in \mathbb{N}.$$

Set

$$E_q^- := \min(\partial\sigma(H^-) \cap [2bq + M_-, 2bq]), \quad E_q^+ := \max(\partial\sigma(H^+) \cap [2bq, 2bq + M_+]).$$

The standard characterization of the almost sure spectrum (see also [6, Theorem 5.35]) yields

$$\Sigma = \bigcup_{q=0}^{\infty} [E_q^-, E_q^+], \quad E_q^- < E_q^+$$

i.e. Σ is represented as a disjoint union of compact intervals, and each interval $[E_q^-, E_q^+]$ contains exactly one Landau level $2bq$. Actually, one has either $E_q^- = 2bq$ or $E_q^+ = 2bq$; more precisely $E_q^- = 2bq$ if $\omega_- = 0$ and $E_q^+ = 2bq$ if $\omega_+ = 0$.

In Theorem 2.1 of [5], the authors describe the behavior of $\mathcal{N}(2bq + E) - \mathcal{N}(2bq)$ when E tends to 0 while in Σ . Under the assumption that u does not decay as fast as in assumption \mathbf{H}_1 , they compute the logarithmic asymptotics of the IDS near $2bq$. Under assumption \mathbf{H}_1 , the authors obtained the optimal logarithmic upper bound and a lower bound that they deemed not to be optimal. In our main result, we obtain the optimal lower bound, thus, proving the logarithmic asymptotics.

Theorem 1. *Let $b > 0$ and assumptions $\mathbf{H}_1 - \mathbf{H}_4$ hold. Assume that, for some $C > 0$ and $\kappa > 0$,*

$$(1) \quad \mathbb{P}(|\omega_0| \leq E) \sim CE^\kappa, \quad E \downarrow 0.$$

Then, for any $q \in \mathbb{N}$, one has

$$(2) \quad \lim_{\substack{E \rightarrow 0 \\ E \in \Sigma}} \frac{\ln |\ln (\mathcal{N}(2bq + E) - \mathcal{N}_b(2bq))|}{\ln |\ln E|} = 2.$$

Thus, Theorem 1 states that, at the Landau level $2bq$, when it is a spectral edge for H_0 , the IDS decays roughly as $e^{-\log^2 |E - 2bq|}$. This decay is faster than any power of $|E - 2bq|$. This explains why we name this behavior also Lifshitz tails even though it is much slower than the Lifshitz tails obtained when the magnetic field is absent (see e.g. [6]).

In [5], the upper bound in (2) is proved under less restrictive assumptions; indeed, Theorem 5.1 of [5] states in particular that, under our assumptions,

$$\limsup_{E \downarrow 0} \frac{\ln |\ln (\mathcal{N}(2bq + E) - \mathcal{N}_b(2bq))|}{\ln |\ln E|} \leq 2.$$

So it suffices to prove

$$(3) \quad \liminf_{E \downarrow 0} \frac{\ln |\ln (\mathcal{N}(2bq + E) - \mathcal{N}_b(2bq))|}{\ln |\ln E|} \geq 2.$$

The improvement over the results in [5] is obtained through a different analysis that borrows ideas and estimates from [1]. The basic idea is to show that, for energies at a distance at most E from $2bq$, the single site potential u can be replaced by an effective potential that has a support of size approximately $|\log E|^{1/2}$ (see section 2 and Lemma 3 therein). This can then be used to estimate the probability of the occurrence of such energies.

1. PERIODIC APPROXIMATION

Assume that hypotheses $\mathbf{H}_1 - \mathbf{H}_4$ hold. For the sake of definiteness, from now on, we assume that $\omega_- = 0$. So, for $q \in \mathbb{N}$, we have $E_q^- = 2bq$. Moreover, (1) becomes $\mathbb{P}(0 \leq \omega_0 \leq E) \sim CE^\kappa$ for $E > 0$ small and $\mathbb{P}(0 \geq \omega_0 \geq -E) = 0$ for any $E > 0$. Up to obvious modifications, the case $\omega_+ = 0$ is dealt with in the same way.

We now recall some useful results from [5]. Pick $a > 0$ such that $\frac{ba^2}{2\pi} \in \mathbb{N}$. Set $L := (2n+1)a/2$, $n \in \mathbb{N}$, and define the random $2L\mathbb{Z}^2$ -periodic potential

$$(4) \quad V_{L,\omega}^{\text{per}}(x) = V_{L,\omega}^{\text{per}}(x) := \sum_{\gamma \in 2L\mathbb{Z}^2} (V_\omega \mathbf{1}_{\Lambda_{2L}})(x + \gamma), \quad x \in \mathbb{R}^2.$$

For $q \in \mathbb{N}$, let Π_q be the orthogonal projection onto the $(q+1)$ -st Landau level i.e. the orthogonal projection onto $\text{Ker}(H_0 - 2bq)$. Consider the bounded operator $\Pi_q V_{L,\omega}^{\text{per}} \Pi_q$. It is invariant by the Abelian group of magnetic translations generated by $2L\mathbb{Z}^2$ (see section 2 in [5]). Hence, $\Pi_q V_{L,\omega}^{\text{per}} \Pi_q$ admits an integrated density of states that we denote by $\rho_{q,L,\omega}(E)$ (see [5]). In [5], we have proved

Theorem 2 ([5]). *Assume that hypotheses $\mathbf{H}_1 - \mathbf{H}_4$ hold and $\omega_- = 0$. Pick $q \in \mathbb{N}$ and $\eta > 0$. Then, there exist $\nu > 0$, $C > 1$ and $E_0 > 0$, such that for each $E \in (0, E_0)$ and $L \geq E^{-\nu}$, we have*

$$\begin{aligned} \mathbb{E}(\rho_{q,L,\omega}(E/C)) - e^{-E^{-\eta}} &\leq \mathcal{N}(2bq + E) - \mathcal{N}(2bq) \\ &\leq \mathbb{E}(\rho_{q,L,\omega}(CE)) + e^{-E^{-\eta}}. \end{aligned}$$

As $\rho_{q,L,\omega}(E)$ is the IDS of the periodic operator $\Pi_q V_{L,\omega}^{\text{per}} \Pi_q$ at energy E , it vanishes if and only if $\sigma(\Pi_q V_{L,\omega}^{\text{per}} \Pi_q) \cap (-\infty, E] \neq \emptyset$. Moreover, $\rho_{q,L,\omega}(E)$ is bounded by CL^d where the constant C is locally uniform in E (see [5]). Thus, we get that, for some $C > 0$,

$$(5) \quad \mathbb{E}(\rho_{q,L,\omega}(E)) \leq CL^d \mathbb{P}\left(\sigma(\Pi_q V_{L,\omega}^{\text{per}} \Pi_q) \cap (-\infty, CE] \neq \emptyset\right).$$

Then, the estimate (3) and, thus, Theorem 1, is a consequence of

Theorem 3. *For $\eta \in (0, 1)$, there exists $C_\eta > 0$ such that, for E sufficiently small and $L \geq 1$, one has, for almost all ω ,*

$$(6) \quad e^{|\log E|^{1-\eta} \log |\log E|} \Pi_q V_{L,\omega}^{\text{per}} \Pi_q \geq \left[\inf_{\gamma \in \Lambda_{2L} \cap \mathbb{Z}^2} \left(\sum_{|\beta - \gamma| \leq |\log E|^{(1-\eta)/2}} \omega_\beta \right) - e^{-|\log E|^{1-\eta}/C_\eta} \right] \Pi_q.$$

The proof of Theorem 3 relies on Lemma 3 which shows that, at the expense of a small error in energy, we can “enlarge” the support of the single site potential u . Lemma 3 is stated and proved in section 2.

Let us now use Theorem 3 to complete the proof of (3) and, thus, of Theorem 1. Pick $L \asymp E^{-\nu}$, ν given by Theorem 2 and fix $\eta \in (0, 1)$ arbitrary.

Thus, by Theorem 2, (5) implies that, for $E > 0$ small,

$$(7) \quad \mathcal{N}(2bq + E) - \mathcal{N}(2bq) \leq CL^d \mathbb{P} \left(\sigma(\Pi_q V_{L,\omega}^{\text{per}} \Pi_q) \cap (-\infty, CE] \neq \emptyset \right) + e^{-E^{-\eta}}.$$

Using (6), as the random variables $(\omega_\gamma)_{\gamma \in \mathbb{Z}^2}$ are i.i.d., for $E > 0$ small, we compute

$$(8) \quad \begin{aligned} & \mathbb{P} \left(\sigma(\Pi_q V_{L,\omega}^{\text{per}} \Pi_q) \cap (-\infty, CE] \neq \emptyset \right) \\ & \leq \mathbb{P} \left(\inf_{\gamma \in \Lambda_{2L} \cap \mathbb{Z}^2} \left[\sum_{|\beta - \gamma| \leq |\log E|^{(1-\eta)/2}} \omega_\beta \right] - e^{-|\log E|^{1-\eta}/C_\eta} \leq e^{-|\log E|/2} \right) \\ & \leq CL^d \mathbb{P} \left(\sum_{|\beta| \leq |\log E|^{(1-\eta)/2}} \omega_\beta \leq 2e^{-|\log E|^{1-\eta}/C_\eta} \right). \end{aligned}$$

Recall that, by (1), as $\omega_- = 0$, one has $\mathbb{P}(0 \leq \omega_0 \leq E) \sim CE^\kappa$ and $\mathbb{P}(0 \geq \omega_0 \geq -E) = 0$ for $E > 0$ small. Hence, by a classical standard large deviation result (see e.g. [2]), we obtain that

$$\mathbb{P} \left(\sum_{|\beta| \leq |\log E|^{(1-\eta)/2}} \omega_\beta \leq 2e^{-|\log E|^{1-\eta}/C_\eta} \right) \leq C_\eta e^{-|\log E|^{2-2\eta}/C_\eta}.$$

Thus, as $L \asymp E^{-\nu}$, this, (7) and (8) yield, for $E > 0$ small,

$$\mathcal{N}(2bq + E) - \mathcal{N}(2bq) \leq C_\eta e^{-|\log E|^{2-2\eta}/C_\eta}.$$

As this bound holds for any $\eta > 0$, we obtain (3) and, thus, complete the proof of Theorem 1. \square

2. THE PROOF OF THEOREM 3

Recall that, for $q \in \mathbb{N}$, Π_q is the orthogonal projection on the eigenspace of H_0 corresponding to $2bq$, the $(q+1)$ -st Landau level of H_0 . We recall

Lemma 1 ([7]). *Pick $p > 1$ and let $V \in L^p(\mathbb{R}^2)$ be radially symmetric. Let $(\mu_{q,k}(V))_{k \in \mathbb{N}}$ be the eigenvalues of the compact operator $\Pi_q V \Pi_q$ repeated according to multiplicity. Then, for $k \in \mathbb{N}$, one has*

$$\mu_{q,k}(V) = \langle V \varphi_{q,k}, \varphi_{q,k} \rangle$$

where

- the functions $\varphi_{q,k}$ are given by

$$\varphi_{q,k}(x) := \sqrt{\frac{q!}{\pi k!}} \left(\frac{b}{2} \right)^{(k-q+1)/2} (x_1 + ix_2)^{k-q} L_q^{(k-q)}(b|x|^2/2) e^{-b|x|^2/4},$$

for $x = (x_1, x_2) \in \mathbb{R}^2$,

- $L_q^{(k-q)}$ are the generalized Laguerre polynomials given by

$$L_q^{(k-q)}(\xi) := \sum_{l=\max\{0, q-k\}}^q \binom{k}{q-l} \frac{(-\xi)^l}{l!}, \quad \xi \geq 0, \quad q \in \mathbb{N}, \quad k \in \mathbb{N},$$

- $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\mathbb{R}^2)$.

Finally, for $k \in \mathbb{N}$, a normalized eigenfunctions of $\Pi_q V \Pi_q$ corresponding to the eigenvalue $\mu_{q,k}(V)$ is equal to $\varphi_{q,k}$. In particular, the eigenfunctions are independent of V .

We denote by $D(x, R)$ the disk of radius $R > 0$, centered at $x \in \mathbb{R}^2$. We set $\nu_{q,k}(R) := \mu_{q,k}(\mathbf{1}_{D(0,R)})$ where $\mathbf{1}_A$ is the characteristic function of the set A .

Lemma 2. Fix $q \in \mathbb{N}$. Define $\varrho = \varrho(R) := bR^2/2$ and

$$(9) \quad \nu_{q,k}^0(R) = \frac{e^{-\varrho} \varrho^{-q+1+k}}{q!} \frac{(k-\rho)^{2q-1}}{k!}.$$

Pick $\beta \in (0, 2)$. Let $f : [1, +\infty) \rightarrow [1, +\infty)$ be such that

$$(10) \quad k^{2q-1} f^{-2q}(k) + k f^{-\beta}(k) \xrightarrow{k \rightarrow +\infty} 0.$$

Then, there exists $k_0 \geq 1$ and $C > 0$ such that, for $k \geq k_0$,

$$(11) \quad \sup_{\substack{R > 0 \\ \varrho(R) \leq k-f(k)}} \left| \frac{\nu_{q,k}(R)}{\nu_{q,k}^0(R)} - 1 \right| \leq C \left(\frac{k^{2q-1}}{f^{2q}(k)} + \frac{k}{f^{\beta+1}(k)} \right).$$

This lemma is an extension of Corollary 2 in [1] to a larger range of radii R .

Proof of Lemma 2. By Lemma 1, passing to polar coordinates (r, θ) in the integral $\langle \mathbf{1}_{D(0,R)} \varphi_{q,k}, \varphi_{q,k} \rangle$ and changing the variable $br^2/2 = \xi$, the eigenvalues $\nu_{q,k}(R)$ of the operator $\Pi_q \mathbf{1}_{D(0,R)} \Pi_q$ are written as

$$\nu_{q,k}(R) = \frac{q!}{k!} \int_0^\varrho \xi^k \left[L_q^{(k-q)}(\xi) \right]^2 e^{-\xi} d\xi.$$

For $q = 0$, we have

$$(12) \quad \nu_{0,k}(R) = \frac{1}{k!} \int_0^\varrho \xi^k e^{-\xi} d\xi = \frac{e^{-\varrho} \varrho^{k+1}}{k!} \int_0^1 e^{\rho t + k \log(1-t)} dt.$$

Now, using a Taylor expansion at 0 and the concavity of $t \mapsto \rho t + k \log(1-t)$, write

$$\begin{aligned} \int_0^1 e^{\rho t + k \log(1-t)} dt &= \int_0^{(k-\rho)^{-\beta/2}} e^{\rho t + k \log(1-t)} dt + \int_{(k-\rho)^{-\beta/2}}^1 e^{\rho t + k \log(1-t)} dt \\ &= \int_0^{(k-\rho)^{-\beta/2}} e^{-(k-\rho)t} \left(1 + O(k(k-\rho)^{-\beta}) \right) dt \\ &\quad + O\left(e^{-(k-\rho)^{1-\beta/2}}\right) \\ &= \frac{1}{k-\rho} + O\left(\frac{k}{(k-\rho)^{\beta+1}}\right) \end{aligned}$$

This and (12) yields (11) when $q = 0$.

Consider now the case $q \geq 1$. For some $C_q > 0$, one has

$$(13) \quad \forall k \geq 1, \quad \sup_{s \in \{0, \dots, q\}} \left| k^{s-q} \binom{k}{q-s} (q-s)! - 1 \right| \leq \frac{C_q}{k}.$$

In order to check (11), we assume that $k \geq q$. In this case, using (13), we compute

$$(14) \quad \begin{aligned} \nu_{q,k}(R) &= \frac{q!}{k!} \sum_{l,m=0}^q (-1)^{l+m} \frac{1}{m!l!} \binom{k}{q-l} \binom{k}{q-m} \int_0^\varrho e^{-\xi} \xi^{k-q+m+l} d\xi \\ &= V(k, q) + R(k, q) \end{aligned}$$

where

$$(15) \quad \begin{aligned} V(k, q) &= \frac{1}{k!q!} \sum_{l,m=0}^q (-1)^{l+m} \binom{q}{l} \binom{q}{m} k^{2q-l-m} \int_0^\varrho e^{-\xi} \xi^{k-q+m+l} d\xi \\ &= \frac{1}{k!q!} \int_0^\varrho e^{-\xi} \xi^{k-q} (k - \xi)^{2q} d\xi, \end{aligned}$$

and

$$(16) \quad \begin{aligned} |R(k, q)| &\leq \frac{C_q}{k} \frac{1}{k!q!} \sum_{l,m=0}^q \binom{q}{l} \binom{q}{m} k^{2q-l-m} \int_0^\varrho e^{-\xi} \xi^{k-q+m+l} d\xi \\ &\leq \frac{C_q}{k} \frac{1}{k!q!} \int_0^\varrho e^{-\xi} \xi^{k-q} (k + \xi)^{2q} d\xi. \end{aligned}$$

For $\rho \leq k - f(k)$, using (10), one computes

$$(17) \quad \left| \frac{R(k, q)}{V(k, q)} \right| \leq C \frac{k^{2q-1}}{f^{2q}(k)} \xrightarrow{k \rightarrow +\infty} 0.$$

On the other hand, as in the case $q = 0$, we have

$$\int_0^\varrho e^{-\xi} \xi^{k-q} (k - \xi)^{2q} d\xi = e^{-\rho} \rho^{k-q+1} (k - \rho)^{2q} I(k, \rho)$$

where

$$I(k, \rho) = \int_0^1 e^{\rho\xi} (1 - \xi)^{k-q} \left(1 + \frac{\rho}{k - \rho} \xi \right)^{2q} d\xi.$$

The function $t \mapsto \rho t + (k - q) \log(1 - t) + 2q \log \left(1 + \frac{\rho}{k - \rho} t \right)$ is concave on $[0, 1]$ and its derivative at 0 is

$$\rho - k + q + 2q\rho/(k - \rho) = (\rho - k) (1 + O(k(k - \rho)^{-2})).$$

Hence, as in the case $q = 0$, we obtain that

$$I(k, \rho) = \frac{1}{k - \rho} + O\left(\frac{k}{(k - \rho)^{\beta+1}}\right).$$

Plugging this into (15), using (17) and (16), and replacing in (14), we obtain (11) for $q \geq 1$.

This completes the proof of Lemma 2. \square

We will now use Lemma 2 to derive the “enlargement of obstacles” lemma for the Landau-Anderson model; we prove

Lemma 3. *Let $q \in \mathbb{N}$ and fix $b > 0$. Fix $\varepsilon > 0$. There exists $C_0 > 0$ and $R_0 > 1$ such that, for each $R \geq R_0$,*

$$(18) \quad \Pi_q \mathbf{1}_{D(0, \varepsilon)} \Pi_q \geq e^{-C_0 R^2 \log R} \left(\Pi_q \mathbf{1}_{D(0, R)} \Pi_q - e^{-R^2/C_0} \Pi_q \mathbf{1}_{D(0, 2R)} \Pi_q \right).$$

This lemma is basically Lemma 2 in [1] except that we want to control the behavior of the constants coming up in the inequality in terms of R .

Proof of Lemma 3. We fix $\delta \in (0, 1)$. Recall Lemma 2, in particular (11) and (9). Pick $C > 2b$ and set $k_0 = k_0(R) := CR^2$. Let f satisfy (10). Hence, there exists $R_0 > 0$ such that, for $R \geq R_0$ and $k \geq k_0 = k_0(R)$, one has $k - f(k) \geq \rho(R)$. Thus, Lemma 2 implies that, for $\tilde{R} \in [R/2, 2R]$, one has

$$(19) \quad \begin{aligned} (1 - \delta) \frac{e^{-\varrho(\tilde{R})} \varrho(\tilde{R})^{k-q+1}}{q!} \frac{(k - \rho(\tilde{R}))^{2q-1}}{k!} &\leq \nu_{q,k}(\tilde{R}) \\ &\leq (1 + \delta) \frac{e^{-\varrho(\tilde{R})} \varrho(\tilde{R})^{k-q+1}}{q!} \frac{(k - \rho(\tilde{R}))^{2q-1}}{k!}. \end{aligned}$$

We show that, if $R \geq R_0$, then, the operator inequality

$$(20) \quad \Pi_q \mathbf{1}_{D(0,\varepsilon)} \Pi_q \geq C_1 (\Pi_q \mathbf{1}_{D(0,R)} \Pi_q - C_2 \Pi_q \mathbf{1}_{D(0,2R)} \Pi_q)$$

holds with the following constants:

$$(21) \quad C_1 := \min_{k \in \{0, \dots, k_0\}} \frac{\nu_{q,k}(\varepsilon)}{\nu_{q,k}(R)} \geq \frac{1}{C_0} e^{-2CR^2 \log R};$$

the lower bound holds for sufficiently large R and, as $k_0 = CR^2$, is a consequence of (9) and (11) written for $\nu_{q,k}(\varepsilon)$;

$$(22) \quad C_{2,q} := \frac{1 + \delta}{1 - \delta} \left(\frac{C}{C - 2b} \right)^{2q-1} 2^{-2(k_0-q+1)} e^{-\varrho(R) + \varrho(2R)} \leq e^{-R^2/C_0};$$

the upper bound holds for sufficiently large R and follows from $k_0 \geq \rho(2R)$.

By Lemma 1, the operators $\Pi_q \mathbf{1}_{D(0,\varepsilon)} \Pi_q$, $\Pi_q \mathbf{1}_{D(0,R)} \Pi_q$, and $\Pi_q \mathbf{1}_{D(0,2R)} \Pi_q$, are reducible in the same basis $\{\varphi_{q,k}\}_{k \in \mathbb{N}}$. Hence, in order to prove (20), it suffices to check that, for each $k \in \mathbb{N}$, the following numerical inequality holds

$$(23) \quad \nu_{q,k}(\varepsilon) \geq C_1 (\nu_{q,k}(R) - C_{2,q} \nu_{q,k}(2R)).$$

If $k \leq k_0$, then (23) holds as $\nu_{q,k}(\varepsilon) \geq C_1 \nu_{q,k}(R)$ by (21). As $k_0 \leq CR^2$ for $C > 2b$ and $\rho = bR^2/2$, or $k \geq k_0$, one has $C(k - \rho(2R)) \geq (C - 2b)(k - \rho(R))$. Thus, by (19) and (22), we have

$$\begin{aligned} \nu_{q,k}(R) - C_{2,q} \nu_{q,k}(2R) &\leq (1 + \delta) \frac{e^{-\varrho(R)} \varrho(R)^{-q+1}}{q!} \frac{(k - \rho(R))^{2q-1} \varrho(R)^k}{k!} \\ &\quad - \left(\frac{1 + \delta}{1 - \delta} 2^{-2(k_0-q+1)} e^{-\varrho(R) + \varrho(2R)} \left(\frac{C(k - \rho(2R))}{(C - 2b)(k - \rho(R))} \right)^{2q-1} \right. \\ &\quad \left. \times (1 - \delta) \frac{e^{-\varrho(2R)} \varrho(2R)^{-q+1}}{q!} \frac{(k - \rho(R))^{2q-1} \varrho(2R)^k}{k!} \right) \\ &= (1 + \delta) \frac{e^{-\varrho(R)}}{q!} \frac{(k - \rho(R))^{2q-1} \varrho(2R)^{k-q+1}}{k!} 2^{2(q-1)} (2^{-2k} - 2^{-2k_0}). \end{aligned}$$

Hence, we find that $\nu_{q,k}(R) - C_2\nu_{q,k}(2R) \leq 0$ if $k \geq k_0$, which again implies (23). This completes the proof of Lemma 3. \square

We now prove Theorem 3.

The magnetic translations for the constant magnetic field problem in two-dimensions are defined as follows (see e.g. [8]). For any field strength $b \in \mathbb{R}$, any vector $\alpha \in \mathbb{R}^2$, the magnetic translation by α , say, U_α^b is defined as

$$U_\alpha^b f(x) := e^{\frac{ib}{2}(x_1\alpha_2 - x_2\alpha_1)} f(x + \alpha) \quad f \in C_0^\infty(\mathbb{R}^2).$$

The invariance of H_0 with respect to the group of magnetic translations $(U_\alpha^b)_{\alpha \in \mathbb{Z}^2}$ implies that, for $\gamma \in \mathbb{Z}^2$, one has

$$(24) \quad U_\gamma^b \Pi_q \mathbf{1}_{D(0,\varepsilon)} \Pi_q U_{-\gamma}^b = \Pi_q \mathbf{1}_{D(\gamma,\varepsilon)} \Pi_q.$$

Hypothesis \mathbf{H}_1 on the single-site potential u guarantees that there exists $\varepsilon \in (0, 1/2)$ so that $V_\omega \geq \sum_{\gamma \in \mathbb{Z}^2} \omega_\gamma \mathbf{1}_{D(\gamma,\varepsilon)}$. Plugging this into (4), we get

$$(25) \quad V_{L,\omega}^{\text{per}} \geq \sum_{\gamma \in 2L\mathbb{Z}^2} \sum_{\beta \in \Lambda_{2L} \cap \mathbb{Z}^2} \omega_\beta \mathbf{1}_{D(\gamma+\beta,\varepsilon)}.$$

Fix $\eta \in (0, 1)$ and pick $R \asymp |\log E|^{(1-\eta)/2}$. Lemma 3 and (24) imply that

$$\Pi_q \mathbf{1}_{D(\gamma,\varepsilon)} \Pi_q \geq e^{-C_0 R^2 \log R} \left(\Pi_q \mathbf{1}_{D(\gamma,R)} \Pi_q - e^{-R^2/C_0} \Pi_q \mathbf{1}_{D(\gamma,2R)} \Pi_q \right).$$

Hence, as the random variables $(\omega_\gamma)_{\gamma \in \mathbb{Z}^2}$ are bounded, this and (25) imply that

$$\begin{aligned} e^{C_0 R^2 \log R} \Pi_q V_{L,\omega}^{\text{per}} \Pi_q &\geq e^{C_0 R^2 \log R} \sum_{\gamma \in 2L\mathbb{Z}^2} \sum_{\beta \in \Lambda_{2L} \cap \mathbb{Z}^2} \omega_\beta \Pi_q \mathbf{1}_{D(\gamma+\beta,\varepsilon)} \Pi_q \\ &\geq \sum_{\gamma \in 2L\mathbb{Z}^2} \sum_{\beta \in \Lambda_{2L} \cap \mathbb{Z}^2} \omega_\beta \Pi_q \mathbf{1}_{D(\gamma+\beta,R)} \Pi_q \\ &\quad - C e^{-R^2/C_0} \sum_{\gamma \in 2L\mathbb{Z}^2} \sum_{\beta \in \Lambda_{2L} \cap \mathbb{Z}^2} \Pi_q \mathbf{1}_{D(\gamma+\beta,2R)} \Pi_q \\ &\geq \Pi_q \sum_{\gamma \in 2L\mathbb{Z}^2} \sum_{\beta \in \Lambda_{2L} \cap \mathbb{Z}^2} \omega_\beta \sum_{|\nu-\gamma-\beta| \leq R/2} \mathbf{1}_{|x-\nu| \leq 1/2} \Pi_q - C e^{-R^2/C_0} R^2 \Pi_q \\ &\geq \left(\inf_{\gamma \in \Lambda_{2L} \cap \mathbb{Z}^2} \left(\sum_{|\beta-\gamma| \leq R/2} \omega_\beta \right) - C R^2 e^{-R^2/C_0} \right) \Pi_q \end{aligned}$$

Taking into account $R \asymp |\log E|^{(1-\eta)/2}$, this completes the proof of Theorem 3.

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